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PHYSICAL MECHANISMS FOR SPORADIC WIND WAVE HORSE-SHOE PATTERNS

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Abstract – We consider three-dimensional crescent-shaped patterns often seen on water surface in natural basins and observed in wave tank experiments. The most common of these 'horse-shoe-like' patterns appear to be sporadic, *i.e.*, emerging and disappearing spontaneously even under steady wind conditions. The paper suggests a qualitative model of these structures aimed at explaining their sporadic nature, physical mechanisms of their selection and their specific asymmetric form.

Introduction

Patterns of spectacular crescent-shaped form, resembling horse-shoes, can be often seen on water surface in natural basins. These patterns are very important from the ocean science perspective, since they modify the airflow above the surface and thus affect the air-sea momentum transfer. They also change in a specific way the radar scattering from the sea surface, and, last but not least, they require conceptually new models to describe statistically the wind-wave field dynamics in their presence [11]. On the other hand, the phenomenon is of true interest from the viewpoint of nonlinear science, as a new non-trivial scenario of pattern formation having no close analogues in the very rich literature on patterns (*e.g.* [3]).

Though no field studies of the horse-shoe patterns are known, among their main features inferred from common observations are [11]:

(i) wave fronts are of crescent-shaped form and are *always oriented forward*,

(ii) the patterns appear quickly after the onset of a fresh wind, at early stages of wave development characterized by high wave steepness;

(iii) the patterns might be localized in space and time, but exist over space- and timescales greatly exceeding the basic wave wavelength and period.

A number of experiments carried out in wave tanks ([4,13]), and wind-wave facilities ([4,8,10]), both in the presence and absence of wind, made it possible to reproduce the patterns in their ideal form under controlled conditions and to measure their parameters. It has been established that the inception mechanism of the horse-shoes is due to McLean's class II instability (five-wave decay) of the plane basic wave. It was also found that wave steepness above a certain threshold is necessary for their inception. However, the specific mechanisms of the pattern formation remain unidentified.

A possible scenario of the emergence of the patterns was suggested in [11]. The authors made the assumption that the *single symmetric pair* with the largest linear growth rate due to McLean's class II instability prevails, while all other pairs within the instability domain may be ignored. Consideration of the simple three-wave problem comprising a basic wave and the fastest growing pair of satellites demonstrated the formation of patterns that indeed render the specific geometry closely resembling the observed horse-shoe forms, but only *under a* *special relation* between the phases of the interacting waves. However, in the conservative case, no preference for phases was found. Instead, the phase of a satellite during the cycle of its growth and subsequent decay rotates through all the possible values, giving a full range of various three-dimensional patterns of all orientations with overall distribution being symmetric in up- and downwind directions, in an apparent contradiction to the observations. However, it was shown that non-conservative effects, inserted into the system but small enough to preserve the Hamiltonian structure to the required order, allow the system to evolve to a steady state with the geometric form of the free surface with the required characteristics.

The model of [11] used the rather crude assumption (based on the experimental evidence [4]) that the class II instability of the fundamental can be adequately represented by the selection of a single symmetric pair of oblique satellites. Although the domain of the five-wave instability is $O(\varepsilon^3)$ narrow (ε being the measure of the basic wave steepness), it is still *continuous* and there is no *a priori* reason to confine the consideration of this instability to a single pair of satellite harmonics. Moreover, this approach completely neglects the sideband satellites growing due to the Benjamin-Feir instability. On the other hand, from the experimental viewpoint, the most often observed crescent-shaped patterns are *sporadic* and seem to be far from equilibrium. Besides that, the patterns have been also observed in the tank experiments *without wind* (*e.g.* [13]) where no equilibria are possible. All this prompts us to reexamine the basic assumptions of [11] and to look for a new robust mechanism able to create the *essentially non-stationary*, relatively *short-lived* patterns of the same geometry.

We consider the general situation when the class II instability gives rise to *many pairs* of oblique satellites, with Benjamin-Feir (class I) satellites being also taken into account. Since these questions constitute a problem evidently not tractable by analytical means, an extensive numerical study of possible evolution scenarios was undertaken. For this purpose, a novel numerical method recently developed by the authors [1] has been used. This method, based on the perturbed integrodifferential Zakharov equation, allows one to trace the long-term evolution of a sufficiently large number of interacting modes, both in a conservative system and in the presence of weak non-conservative effects included via the insertion of small (of the order of the five-wave interaction term) linear damping and forcing terms.

The numerical analysis allowed us to establish the basic facts concerned with the long-time evolution of five-wave instability. First, even the system with the initially large number of degrees of freedom appears to follow, at least locally, the evolution scenario of a low-dimensional system. This means that in a generic situation only very few of the linearly unstable modes (often different modes at different moments of time) would grow considerably. Moreover, under the effect of weak dissipation this small number of grown modes is normally reduced to just single pair. This suggests the existence of a nonlinear *selection mechanism* which is enhanced by dissipation. This mechanism reduces considerably the number of grown modes, at the same time noticeably enlarging their maximal amplitude. Evolution of the system is shown to result in the formation of *sporadic* crescent-shaped patterns on the free surface with steep front slopes and flattened rear ones, closely resembling the observed horse-shoe patterns. The patterns' dynamics is found to be determined mainly by quintet interactions, while the presence of Benjamin-Feir modulation does not alter it qualitatively.

The selection mechanism has been identified as follows: each growing pair of oblique satellites 'pushes' all other pairs out of the narrow instability domain by creating nonlinear frequency shift. Normally, one pair prevails, so that most of the time the evolution closely corresponds to that of the three-wave system. At the same time, the presence of other pairs of satellites is shown to lead to *phase asymmetry*: a growing satellite pair has the required ('correct') phase value, while the phases of decaying ones are unstable, so that the frequency shifts caused by the weak interaction with other modes set them into rapid rotation. As a result, pronounced horse-shoe like patterns develop at each cycle of such intermittent regime. The described mechanism does not require the presence of non-conservative effects as a prerequisite. Nevertheless, weak dissipation is shown to significantly facilitate observation of the patterns, due to prolonged satellites' growth and elimination of noise.

1. Basic equations and the numerical model

We consider three-dimensional potential gravity waves on the free surface of an incompressible fluid of infinite depth. Wave slopes are supposed to be of the order of a small parameter ε. Dynamics then is governed, up to the order ε^4, by the integrodifferential equation

$$\begin{aligned} i\frac{\partial b_0}{\partial t} &= (\omega_0 + i\gamma_0)b_0 + \int V_{0123} b_1^* b_2 b_3 \delta_{0+1-2-3}\, d\mathbf{k}_{123} \\ &+ \int W_{01234} b_1^* b_2 b_3 b_4 \delta_{0+1-2-3-4}\, d\mathbf{k}_{1234} + \frac{3}{2}\int W_{43210} b_1^* b_2^* b_3 b_4 \delta_{0+1+2-3-4}\, d\mathbf{k}_{1234} \end{aligned} \tag{1}$$

derived by [15] (see also [6]) and extended to the order ε^4 by [7]; for our purposes, equation (1) is modified by the presence of small (of the order ε^4) non-conservative effects. Here, $b(\mathbf{k})$ is a canonical complex variable, $\omega(\mathbf{k}) = (gk)^{1/2}$ is the linear dispersion relation, $k = |\mathbf{k}|$, $\gamma(\mathbf{k})$ stands for $O(\varepsilon^3)$ damping/growth rate, integration in (1) is performed over the entire $\mathbf{k}$-plane. The compact notation used designates the arguments by indices, *e.g.* $V_{0123} = V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$, $\delta_{0+1-2-3} = \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, $d\mathbf{k}_{123} = d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$, asterisk means complex conjugation, t is time. The canonical variable $b(\mathbf{k})$ is linked to the Fourier-transformed primitive physical variables $\varphi(\mathbf{k}, t)$ and $\eta(\mathbf{k}, t)$ (free-surface potential and surface elevation, respectively) through an integral-power series ([7]). All details of the lengthy procedure of derivation of (1), as well as the expressions for the kernels V, W can be found in [7].

Equation (1) is the so called *reduced equation*, that is, it takes into account only resonant (or nearly resonant) interactions between free modes, leaving incomparably fewer interactions to consider than in conventional physical-space models. In other words, much of the complexity of the original hydrodynamic equations goes to coefficients that indeed have very cumbersome algebraic form. More advantages of (1) as a basis for a numerical study can be pointed out ([2]). Though the idea to use this equation, upon proper discretization, to study the evolution of a number of discrete modes seems natural ([5]), there were few attempts at implementation of such a numerical scheme.

In the present work, the recently proposed novel numerical approach ([1]) is used. Given the initial state of the fluid (functions $\varphi(x, y, 0)$, $\eta(x, y, 0)$), Fourier transformation and integral power series expansion are used to obtain the initial value of complex amplitude $b(\mathbf{k}, 0)$. Then the function b is discretized in $\mathbf{k}$-space, being replaced by a set of complex variables $b_m \equiv b(\mathbf{k}_m)$, $m = 1, 2, \ldots N$. Next, all the coefficients V and W are calculated and stored. For the subsequent integration in time, the discretized version of (1) is used. Finally, the obtained fluid state is transformed back to physical variables.

When the non-conservative effects are negligible for the timescales under consideration, (1) is a Hamiltonian system and time integration is performed with a symplectic algorithm, otherwise the system is integrated by the more conventional Runge-Kutta scheme. All the details of the proposed algorithm are given in [1].

2. Instability of a single basic wave: evolution scenarios

The model of [11] is confined to the consideration of only three harmonics in the Zakharov equation. In reality, of course, neither the basic wave nor the satellites are monochromatic. In this section we will focus upon the latter, *i.e.* we will consider nonlinear evolution of many satellites generated by a single basic wave due to quintet interactions, while the effects of non-monochromaticity of the basic wave are briefly considered in §5.

2.1. *Three-wave model*

Consider a system comprising only three waves ($N = 3$) with the amplitudes a, b, c and wavevectors $\mathbf{k}_a$, $\mathbf{k}_b$, $\mathbf{k}_c$ satisfying the condition $3\mathbf{k}_a = \mathbf{k}_b + \mathbf{k}_c$, where a denotes the basic wave, b, c are the satellites, assumed initially small, $\mathbf{k} = (k_x, k_y)$. Without the loss of generality

$$\mathbf{k}_a = (1, 0), \qquad \mathbf{k}_b = (\frac{3}{2} + p, q), \qquad \mathbf{k}_c = (\frac{3}{2} - p, -q). \tag{2}$$

It is implied that the waves a, b, c form a nearly resonant quintet, that is

$$3\omega_a - \omega_b - \omega_c \leq O(\varepsilon^3). \tag{3}$$

Thus, this combination of waves represents the simplest possible case of quintet interactions. Equation (1) for this case takes the form

$$\begin{aligned} a_t &= -\mathrm{i}(\omega_a + i\gamma_a)a - \mathrm{i}\left[V_{aaaa}|a|^2 + 2V_{abab}|b|^2 + 2V_{acac}|c|^2\right]a - 3\mathrm{i}W_{bcaaa}bc\left(a^2\right)^*, \\ b_t &= -\mathrm{i}(\omega_b + i\gamma_b)b - \mathrm{i}\left[2V_{abab}|a|^2 + V_{bbbb}|b|^2 + 2V_{bcbc}|c|^2\right]b - \mathrm{i}W_{bcaaa}c^*a^3, \\ c_t &= -\mathrm{i}(\omega_c + i\gamma_c)c - \mathrm{i}\left[2V_{acac}|a|^2 + 2V_{bcbc}|b|^2 + V_{cccc}|c|^2\right]c - \mathrm{i}W_{bcaaa}b^*a^3. \end{aligned} \tag{4}$$

Performing the transformation

$$a = A\exp(-\mathrm{i}\alpha), \qquad b = B\exp(-\mathrm{i}\beta), \qquad c = C\exp(-\mathrm{i}\gamma),$$

system (4), for initially equal and symmetric satellites, reduces to the form

$$\begin{aligned} A_t &= 3W_{bcaaa}A^2B^2\sin\Phi + \Gamma_a A, \\ B_t &= -W_{bcaaa}A^3B\sin\Phi + \Gamma_b B, \\ \Phi_t &= \delta + PA^2 + MB^2 + W_{bcaaa}A(9B^2 - 2A^2)\cos\Phi, \end{aligned} \tag{5}$$

where the parameters are

$$\begin{aligned} &\delta = 3\omega_a - \omega_b - \omega_c, \qquad \Gamma_a = \omega_a\gamma_a, \qquad \Gamma_b = \omega_b\gamma_b, \\ &P = 3V_{aaaa} - 2V_{abab} - 2V_{acac}, \qquad M = 6V_{abab} + 6V_{acac} - 4V_{bcbc} - V_{bbbb} - V_{cccc}. \end{aligned}$$

Variable Φ (the phase of the pair of satellites relative to the fundamental) is defined as $\Phi = 3\alpha - \beta - \gamma$; Φ and sometimes $\sin\Phi$ below will be referred to as 'the phase', for brevity.

The specific form of a surface elevation pattern is mainly prescribed by the value of Φ. If $-\pi + 2\pi n < \Phi < 0 + 2\pi n$, the patterns have their convex sides oriented downwind, the best resemblance with the observed patterns being at $\Phi = -\pi/2 + 2\pi n$. For simplicity we will refer to such phases as *negative* and will omit the period $2\pi n$ hereafter. The phases in the range $0 < \Phi < \pi$ correspond to the opposite orientation. However, as was noted by [11], if $\Gamma_a = \Gamma_b = 0$, then (5) is a reversible Hamiltonian system, so that it passes in a symmetric manner through all the values of Φ. Thus, the system shows a variety of three-dimensional patterns with no preference for orientation. This means that a model of horse-shoe patterns cannot be built upon the basis of such a conservative three-wave system.

Still, a closer inspection of this system is useful for further progress in understanding (see [2] for more detailed analysis). Suppose that $\Gamma_a = \Gamma_b = 0$. If the initial amplitudes of the satellites are infinitesimal, then the trajectories of the system (5) are close to the separatrix originating and ending at two stationary saddle points

$$B = 0,\ A = A_0,\ \Phi = \Phi_0 = \pm\cos^{-1}\frac{\delta + PA_0^2}{2WA_0^3}.$$

For definiteness, and also because this ensures the maximal instability ([12]), we choose $\delta = -PA_0^2$, and then $\Phi_0 = \pm\pi/2$. An example of a triad evolution for such a case is presented in figure 1, for the initial value of phase Φ equal to $-\pi/2$. The dynamics is periodic; the phase, rotating counterclockwise, stays most of the time close to the two values $\Phi = \pm\pi/2$ with quite fast transition in between, zero phase corresponding to the point of the deepest modulation. When the satellites are small compared to the fundamental, all the trajectories in (B, Φ) plane are in the vicinity of the saddle points. Since the trajectories in the vicinity of the saddle ($B = 0$, $\Phi = -\pi/2$) are divergent in B, there is a convergence of the trajectories in Φ, while in the neighbourhood of the second saddle ($B = 0, \Phi = \pi/2$) the trajectories converge in B but diverge in Φ. This implies that if we perturb the system the trajectories corresponding to growing satellites will tend to preserve their phase, while those corresponding to decaying satellites will experience large variations of the phase, and thus become unstable.

These simple facts give insight into more complicated situations. Even a small perturbation can change the phase and thus the whole system dynamics near the point $\Phi = +\pi/2$. Thus, the long periods in the

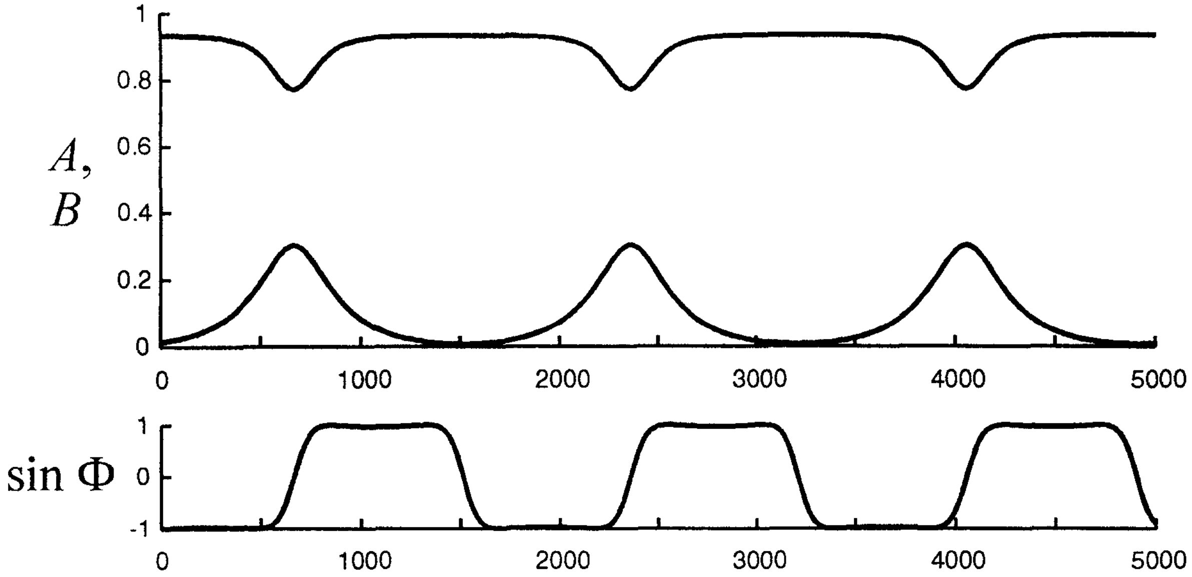

FIGURE 1. Evolution of the conservative three-wave system (2) with $p = 3/2$ and $q = 1.54$, for the basic wave steepness 0.21. Time is measured in ω^{-1}, where ω is the basic wave frequency.

neighbourhood of $\Phi = +\pi/2$ and potential instability of these segments of trajectories indicate a possible *structural instability* of the three-wave system.

With the inclusion of small non-conservative effects (for wind waves, it is natural to assume that $\Gamma_a > 0$, $\Gamma_b < 0$), the similar two saddles are preserved. Numerical simulations (see [2]) show that the evolution becomes slightly asymmetrical since the growth of satellites is slower and their subsequent decay is faster. The point of maximum corresponds to a certain *negative* value of phase. Still, simulations demonstrate that there are still relatively long periods of the phase being close to the potentially unstable value $+\pi/2$, which suggests the non-conservative three-wave system to remain potentially structurally unstable and thus might be too oversimplified to serve as a model of the formation of horse-shoe patterns.

2.2. *Multiple class II satellites*

In reality, a fundamental wave of a finite amplitude possesses finite size *domains* of instability with respect to four- and five-wave processes in the wavevector space (see [9]; [5], Fig. 6.6), so that simultaneous growth of many pairs of satellites (strictly speaking, continuum of satellites) should occur. Nonlinear dynamics of such continuum of linearly growing satellites is simulated by a large number of unstable satellites. Since our primary interest is in three-dimensional dynamics, in this section we focus upon the situations where the satellites were taken in the five-wave (class II) instability domain and stable part of k-plane only. The effect of finite bandwidth of the basic wave spectrum and, in particular, role of the Benjamin-Feir instability are briefly addressed in §5.

For the numerical study, a system comprising a basic wave with wavevector $\mathbf{k}_0 = (1, 0)$ and N pairs of oblique satellites with wavevectors $\mathbf{k}_{j,j+1} = (\frac{3}{2} \pm p_j, \pm q_j)$, $j = 1, 2, \ldots, N$ was selected. In the experiments discussed in this section, $N = 42$, and both symmetric (with respect to k_x-axis) and non-symmetric oblique harmonics within and in the neighbourhood of the instability domain were represented; see [2] for details. Three different values for the initial steepness of the fundamental (0.13, 0.17 and 0.21) were used. The satellites were put initially small, with amplitudes of the order $O(10^{-2})$ relative to the amplitude of the fundamental; no essential dependence on the exact value of satellites' initial amplitude was revealed. Initial phase of all the satellites was again prescribed at the value most favourable for growth ($-\pi/2$). Evolution in time was traced for about 10^3 periods of the basic wave. We again emphasize that at this stage the Benjamin-Feir instability was excluded.

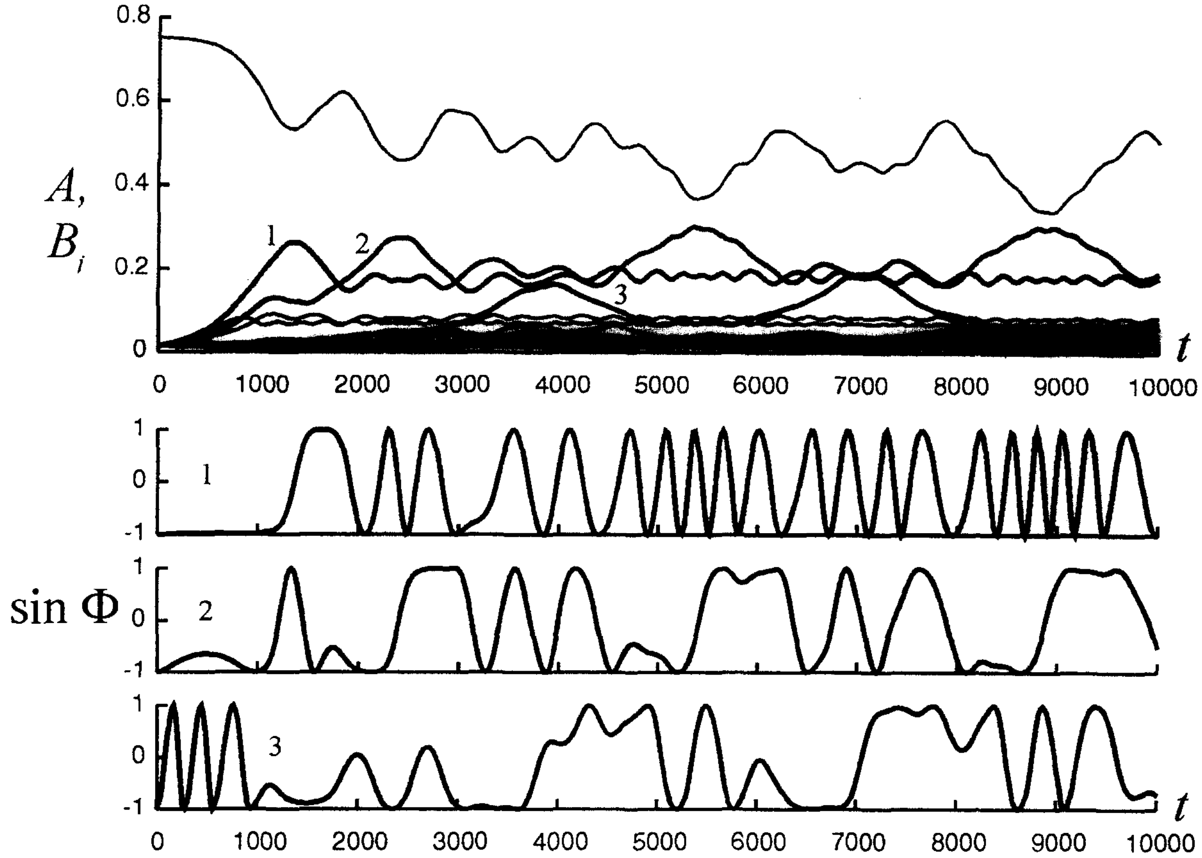

FIGURE 2. Evolution of the conservative system consisting of a basic wave and 42 pairs of initially small satellites. For three most unstable pairs, evolution of phase is shown.

An example of conservative evolution of the system is presented in figure 2. The plot demonstrates two distinct features of the class II instability with respect to multiple satellites. First, most of the modes located in the linear instability domain do not attain considerable amplitudes, stagnating at a quite low level. Only a few modes at each moment can grow inattenuated, though this behaviour is displayed by different modes at different moments. Second, the evolution of the phases of these growing modes differs noticeably from that of the isolated three-wave system. In particular, it becomes essentially asymmetric. When the mode is growing, its phase keeps close to $-\pi/2$, as in the three-mode case; the maximum of the satellite amplitude again corresponds to zero phase, but soon after reaching the maximum of the amplitude, the phase typically starts to change much more rapidly, while the amplitude decreases and tends to stagnate at a quite low level. However, this behaviour is not always well pronounced for all growing pairs during the course of the *conservative* evolution, thus allowing one to consider it only as a tendency.

Meanwhile, these features, observed in the large number of runs of the numerical model, are much more pronounced in the more realistic weakly non-conservative case (figure 3). Only one pair of satellites attains considerable amplitude, while its phase remains in the neighbourhood of $-\pi/2$ nearly *all the time*, except for the brief excursion soon after maximum of amplitude. The above features do not exhibit noticeable dependence on the amplitude of the basic wave provided the growth rates due to inviscid class II instability exceed the dissipation.

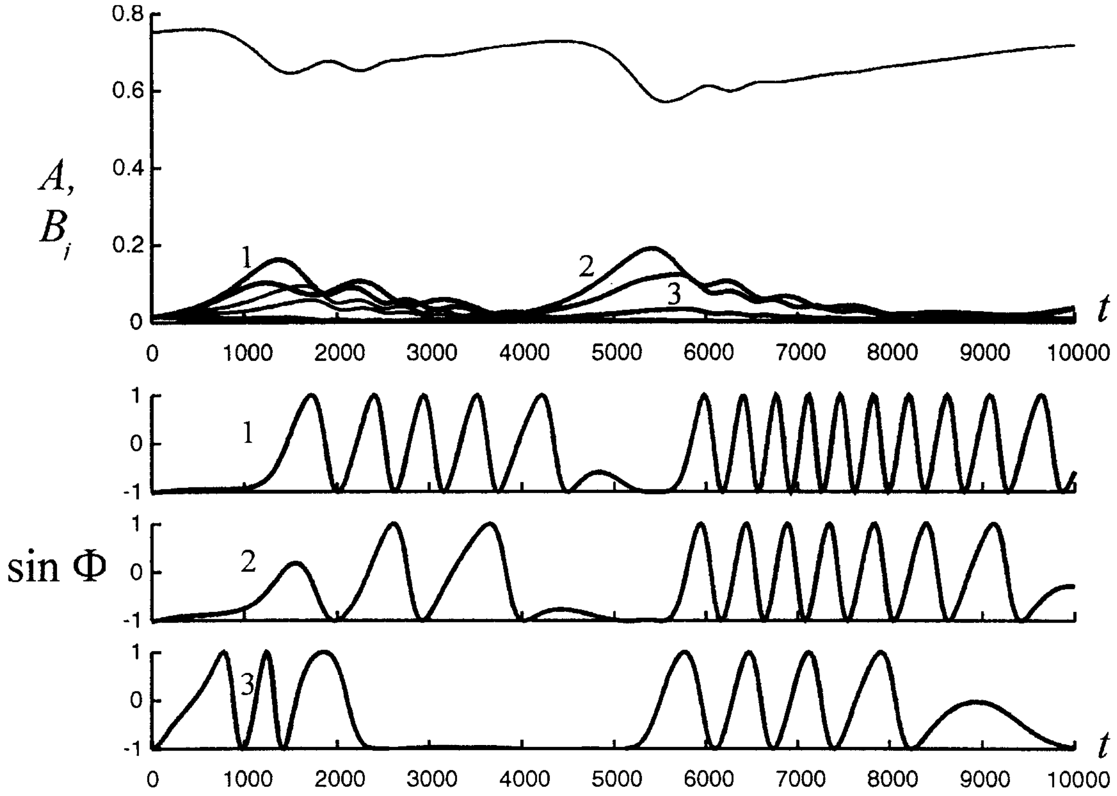

FIGURE 3. As in figure 2, with non-conservative effects included ($\Gamma_a = 5 \cdot 10^{-5}\omega_a$, $\Gamma_j = 5 \cdot 10^{-4}\omega_j$, $j = 1, \dots, N$).

2.3. *Summary and discussion*

While three-wave systems, with small non-conservative effects, exhibit nearly symmetric behaviour with respect to different values of phase, inclusion of one or more additional pairs of satellites drastically enhances the asymmetry. The phase of a satellite during its growth and near the maximum of amplitude remains close to $-\pi/2$; the phase of the decaying one in most cases becomes indeterminate.

The important role of dissipation requires some special comments. First, the dissipation is essential in the initial selection, since only the modes with maximal growth rates and initial phases close to $-\pi/2$ can survive. Second, its presence results in shifting the phase at the point of the satellite maxima: the phase remains negative at the maximum and in its *neighbourhood.* Thus, the phases of modes whose amplitudes exceed a certain threshold, are always negative, and therefore the wave fronts are oriented forward. Such sporadically appearing and disappearing surface patterns closely resemble the observed horse-shoe ones.

The most nontrivial fact established numerically is that the evolution in multi-dimensional system remains *effectively low-dimensional.* It is well known that a nonlinear system characterized at an initial moment by a certain finite number of excited modes can evolve in accordance with two opposite scenarios. If nonlinearity prevails (in a certain sense) over dissipative effects, the number of excited modes of the system tends to grow with time, in the opposite limit the effective number of active modes in the system decreases and the system often falls into a low-modal attractor. At first sight the latter scenario seems to be the case. It is easy to find the rate of contraction of the phase volume ν

$$\nu = \Gamma_a + \sum_{j=1}^{N} \Gamma_{b_j} + \sum_{j=1}^{N} \Gamma_{c_j}$$

and it is indeed negative when the integral dissipation $\left| \sum_{j=1}^{N} \Gamma_{b_j} + \sum_{j=1}^{N} \Gamma_{c_j} \right|$ exceeds the input Γ_a. This is the case for all our simulations when non-conservative effects were taken into account. However, the distinct tendency towards low-dimensional dynamics is revealed even in the case of purely Hamiltonian dynamics! Though this seems to contradict the preservation of the phase volume, the contraction of the phase volume in physically important segments of the phase space is exactly compensated by its expansion in *other* segments. For instance, confinement of the phases of the growing and grown modes is accompanied by the rapid phase rotation of all other modes. If one focuses attention on the subspace of interest, say the modes with amplitudes exceeding a certain small threshold, the rest of phase space will provide a kind of nonlinear dissipation for the chosen subspace.

3. Selection mechanism

Numerical results discussed suggest that there exists a certain nonlinear mechanism responsible for the selection of modes and the eventual formation of horse-shoe patterns, which can be revealed by an analysis of simple low-modal systems.

A monochromatic wave of amplitude a and wavevector $\mathbf{k}_a = (1, 0)$ is known to be unstable with respect to pairs of oblique satellites with wavevectors $\mathbf{k}_b$, $\mathbf{k}_c$ satisfying (2), if a certain condition on p, q implied by (3) is fulfilled. Within the framework of the Zakharov equation, the corresponding condition can be easily obtained in an explicit form [2]:

$$4W_{bcaaa}^2 |\bar{a}|^6 - (\sigma - \mathrm{i}(\gamma_b - \gamma_c))^2 > (\gamma_b + \gamma_c)^2, \tag{6}$$

where $\sigma = \delta + P|\bar{a}|^2$, $\bar{a}$ is the initial value for a. According to such a linearized theory, *all* the initially small oblique satellites in the instability domain grow exponentially, with the rate

$$\mathrm{Re}\left\{ \sqrt{4W_{bcaaa}^2 |\bar{a}|^6 - (\sigma - \mathrm{i}(\gamma_b - \gamma_c))^2 - (\gamma_b + \gamma_c)^2} \right\}.$$

If only one pair is present, its evolution is governed by (5); the process is recurrent, linearized system being valid as an approximation on each cycle in the neighbourhood of the point of the smallest amplitude of the satellites.

The presence of just one extra pair makes the problem much more difficult. However, numerical analysis of the evolution of a system comprising two pairs of satellites typically shows a relatively simple scenario of evolution (detailed analysis is presented in [2]). In most cases the simultaneous growth of both pairs does not occur. Instead, the modes grow in alternance: after the initial linear stage of instability one of harmonics stagnates at a quite low amplitude level, while the other one continues to grow, attains considerable amplitude and decays. At this stage, the presence of the other pair of harmonics does not seem to affect the system behaviour at all, except for a rather remarkable fact that the maximum of amplitude of the growing pair in the presence of the stagnating one is somewhat enlarged. Further on, the process is approximately repeated with the other pair of satellites, so that most of time the evolution appears to be close to that of the three-wave system.

In principle, an approximate solution can be obtained by constructing the asymptotic solution for each stage of the evolution and matching the resulting expansions. Consider, as the simplest non-trivial model, a five-wave system with just two pairs of satellites, using the notation $(\mathbf{k}_b, \mathbf{k}_c)$ for the first pair and $(\mathbf{k}_d, \mathbf{k}_e)$ for the second one, so that $3\mathbf{k}_a = \mathbf{k}_b+\mathbf{k}_c = \mathbf{k}_d+\mathbf{k}_e$, and the frequency mismatches $3\omega_a-\omega_b-\omega_c$ and $3\omega_a-\omega_d-\omega_e$ are both of the order of ε^3. Since our primary interest lies in the selection process, consider the system for the stage $B \simeq D \ll A$, A being the amplitude of the fundamental, B and D of the satellites. Provided that the two pairs lie close to the point of the maximum linear growth rate in k–plane, one may put $W_{bcaaa} \cong W_{deaaa} \equiv W = O(\varepsilon^3)$, while all the other coefficients of the governing system are of the order ε^2. However, in virtue of (6), or its analogue with dissipation taken into account, the total initial frequency mismatch with the nonlinear frequency shifts caused by the basic wave, $\delta_j + P_j A(0)^2$ is of the order of the largest term due to the quintet interaction, *i.e.*, $O(\varepsilon^3)$. Introducing $\tau = Wt$ as a slow timescale, the typical ratio of amplitudes of satellites and the fundamental at the point where selection occurs $B/A \simeq D/A = \mu$ as a new small parameter and retaining only the leading order terms in μ^2, we obtain the system governing the case of two unstable pairs:

$$\begin{aligned}
A_\tau &= 3A^2B^2\sin\Phi_1 + 3A^2D^2\sin\Phi_2 + \hat{\Gamma}_a A,\\
B_\tau &= -A^3B\sin\Phi_1 + \hat{\Gamma}_b B\\
D_\tau &= -A^3D\sin\Phi_2 + \hat{\Gamma}_d D\\
\Phi_{1\tau} &= \frac{[\delta_1 + P_1{A_0}^2]}{W} + \frac{\mu^2\hat{P}_1(A^2 - {A_0}^2)}{\varepsilon} + \frac{\mu^2\hat{N}_1D^2}{\varepsilon} - 2A^3\cos\Phi_1,\\
\Phi_{2\tau} &= \frac{[\delta_2 + P_2{A_0}^2]}{W} + \frac{\mu^2\hat{P}_2(A^2 - {A_0}^2)}{\varepsilon} + \frac{\mu^2\hat{N}_2B^2}{\varepsilon} - 2A^3\cos\Phi_2,
\end{aligned} \tag{7}$$

where $A_0 = A(0)$, $\hat{\Gamma} = \Gamma/W$, $\hat{P}_j$, $\hat{N}_j$ are the new $O(1)$ coefficients.

System (7) enables one to understand the mechanism of the selection process. At the initial stage of evolution, when $B^2/A^2 \ll \mu$ and $D^2/A^2 \ll \mu$, the linear regime is realized: both pairs of satellites grow, provided that Φ_1 and Φ_2 are both negative. When the satellites attain a certain threshold amplitude, the second terms in the right-hand sides of the phase equations in (7) become of order $O(1)$. The signs of these frequency-shifting terms are such that each pair tends to push the phase of the *other* pair from the value which is favourable for the growth of amplitude, until one of the phases starts to rotate, preventing the subsequent growth of the corresponding pair. Afterwards, only one of the pairs grows, while the other one, with the rapidly rotating phase, is decaying. The presence of the small parameter in the denominator of the frequency-shifting terms specifies the characteristic threshold level of the satellite amplitude where the selection occurs: $B \simeq D \simeq \sqrt{\varepsilon}A$. One can describe this mechanism as the 'rivalry' between the pairs, so that each one, while growing, pushes the other one out of the resonance with the fundamental via the frequency shift.

We already mentioned the interesting effect exhibited by the simulations: the maximal amplitude of the remaining pair noticeably *exceeds* that of the same pair in the isolated three-wave system. In other words, small satellites act like a catalyst enhancing the energy exchange between the growing satellite and the fundamental. This fact can be easily explained by comparison of (7) with the single-pair system. The growth of, say, B in (5) is controlled by the change of the amplitude of the fundamental: when A diminishes, the variation of the frequency shift $P_1(A^2 - A_0^2)/W$ becomes large and positive (since $P_1 < 0$), leading to the counterclockwise rotation of phase. Meanwhile, in the corresponding phase equation in (7) $N_1 < 0$, so that the term N_1D^2 is always negative, and thus tends to compensate the effect of the frequency shift for this pair, while the similar term in the other phase equation drives the phase of the second pair out of equilibrium. Hence, the effect tends to stabilize the phase of the first pair at the value that is favourable for growth, eventually resulting in the increase of the maximal amplitude.

4. Effects due to the finite width of the spectrum of the basic wave

An actual wave field is continuous, so that the consideration of the process of interaction of monochromatic waves should be regarded as an approximation to the interaction of finite bandwidth wavepackets. In particular, the basic wave spectrum bandwidth cannot be less than ε. The most important effect in this respect is clearly the presence of the modulational (Benjamin–Feir) instability of the fundamental wavepacket. Does the presence of much faster $(O(\varepsilon)^{-2})$ and more energetic four-wave (class I) processes preserve intact the evolution scenarios

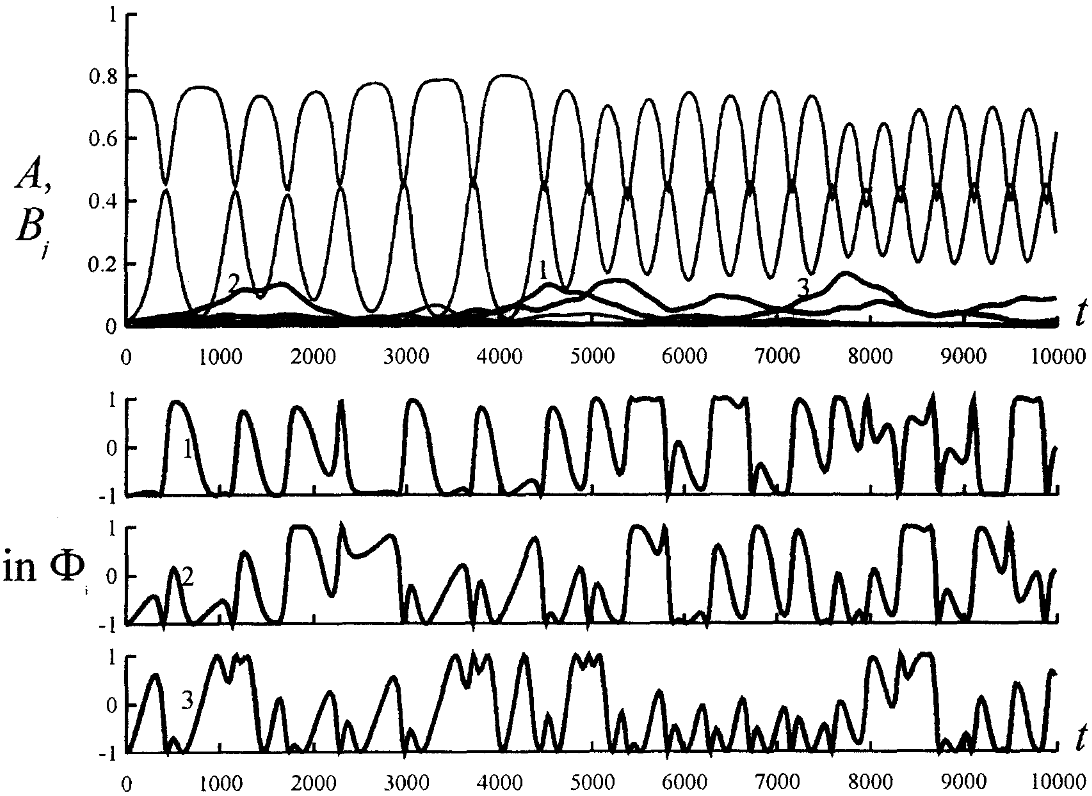

FIGURE 4. As in figure 3, with a pair of Benjamin-Feir satellites included.

established above? As a first approximation, the effect of the modulational instability can be modeled just by adding of one or more pairs of additional initially small satellites $\mathbf{k}_{j\pm} = (1 \pm \Delta_j, 0)$, $j = 1, 2, \ldots$, where $\mathbf{k}_0 = (1, 0)$ is the basic wave central wavevector, Δ_j lie within the domain of the Benjamin–Feir instability (see, *e.g.* [5]).

Based on numerous simulations in wide range of the parameters involved (one sample is given in figure 4) we conclude that *qualitatively*, the account of the finite bandwidth of the basic wave does not alter the already established scenarios of field evolution on ε^{-3}-timescale, although *quantitatively* the effect could be quite noticeable.

It has been previously suggested by [14] that the class I processes, which have shorter characteristic time of development, may trigger the class II instability. Our results do not support this hypothesis. Moreover, in accordance with [12], the presence of the class I instability imposing fast oscillations to the amplitudes and phases of the fundamental and class II satellites appears to *inhibit* five-wave processes. Qualitatively this fact can be easily explained, averaging the equations over fast Benjamin-Feir oscillations: the 'averaged phase' of the satellite will always differ from the optimal one.

5. Discussion

Numerical and analytical consideration carried out above has enabled us to establish a number of properties referring to the formation of three-dimensional wind wave patterns on the free surface.

We have shown that nonlinear evolution of five-wave instability in the generic case can be adequately represented as a low-dimensional process. Though at a linear stage all linearly unstable harmonics do grow simultaneously, most of them cannot attain considerable amplitudes. Instead, at a nonlinear stage the subsequent growth is arrested by the selection mechanism: nonlinear frequency shifts lead in most cases to the competition among the satellites, so that they tend to push each other out of the resonance zone. Since this zone is quite narrow, eventually only a small number (typically, a single pair) of satellites survives.

Existence of this mechanism leads to several important consequences. First, it means that the developed class II instability remains essentially low-modal, at least at characteristic times of the order of modulation cycle. This greatly simplifies the theoretical study of the five-wave decay of a basic wave, being also in agreement with observations ([4]). Second, the harmonics pushed out by a growing satellite at some value of their amplitude (that can be easily estimated) take part in the formation of noise, thus allowing one to introduce the concept of *natural noise level.* Third, the mechanism controls the effective phase of a growing satellite in such a way that it remains negative for growing and grown satellites and is *indeterminate* (rapidly changing) for decaying ones. This gives a scenario for the emergence of horse-shoe patterns on the water surface: at each moment a certain pair of harmonics grows with a fixed phase corresponding to the orientation of the patterns downwind, while the phases of all other satellites are rapidly rotating. According to this scenario, the formation of patterns occurs rather rapidly, with the characteristic time of the class II instability and each particular 'individual modulation' exists for a about the same characteristic time period. This correctly describes the character of the observed sporadic patterns. Moreover, the scenario explains the long remaining enigmatic fact of horse-shoe pattern observations even in the tanks without wind: given the finite size of all installations just a fraction of the first modulation cycle is typically available. Furthermore, in full agreement with the tank observations, the mechanism works only for the steepness of the basic wave exceeding a certain level. Such a threshold character of the process is clearly attributed to the rate of dissipation. It is important to note, however, that apart from this threshold, no essential qualitative dependence of dynamics on the amplitude of the basic wave was noted. The role of the dissipation is more diverse. It creates the important phase shift of grown modes, making the phase at the maxima of the satellite amplitude negative, and, besides that, strongly enhances the phase dynamics asymmetry by prolonging growth and shortening sharply decay time of the satellites. However, steep gravity waves under the action of wind are likely to develop breakers which obviously require much more sophisticated description of nonlinear non-conservative mechanisms compared to the extremely simple generation/dissipation model used in this work. Account of nonlinear non-conservative mechanisms will certainly enrich the dynamics, nevertheless, we expect that qualitatively the basic physics established here will remain intact.

Thus, the simple model considered gives a qualitative description of the sporadic horse-shoe phenomenon which is consistent with the observations.

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